

EXTERIOR DEPTH AND EXTERIOR GENERIC ANNIHILATOR NUMBERS

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ABSTRACT. We study the exterior depth of an E -module and its exterior generic annihilator numbers. For the exterior depth of a squarefree E -module we show how it relates to the symmetric depth of the corresponding S -module and classify those simplicial complexes having a particular exterior depth in terms of their exterior algebraic shifting. We define exterior annihilator numbers analogously to the annihilator numbers over the polynomial ring introduced by Trung and Conca, Herzog and Hibi. In addition to a combinatorial interpretation of the annihilator numbers we show how they are related to the symmetric Betti numbers and the Cartan-Betti numbers, respectively. We finally conclude with an example which shows that neither the symmetric nor the exterior generic annihilator numbers are minimal among the annihilator numbers with respect to a sequence.

1. INTRODUCTION

Let $S = K[x_1, \dots, x_n]$ be the symmetric algebra and $E = K\langle e_1, \dots, e_n \rangle$ denote the exterior algebra over an infinite field K . Let Δ be a simplicial complex on the vertex set $[n] = \{1, \dots, n\}$. Instead of studying the properties of Δ directly one often studies the *Stanley-Reisner ring* $K[\Delta] = S/I_\Delta$, or the *exterior face ring* $K\{\Delta\} = E/J_\Delta$ of Δ . Several algebraic and homological invariants of $K[\Delta]$ over S are analogous to invariants of $K\{\Delta\}$ over E . In this paper we study relations between the corresponding symmetric and exterior invariants, also in the more general context of squarefree modules.

Yanagawa introduced squarefree modules over the polynomial ring as a generalization of squarefree monomial ideals in [Yan00, Definition 2.1]. Römer defined in [Röm01, Definition 1.4] the corresponding notion of a squarefree E -module and showed that there exists an equivalence of categories between the category of squarefree S -modules and the category of squarefree E -modules (see Section 2 for details). A typical example of a squarefree S -module is the Stanley-Reisner ring $K[\Delta]$, of a squarefree E -module the exterior face ring $K\{\Delta\}$. Under the equivalence of categories $K[\Delta]$ corresponds to $K\{\Delta\}$.

Let \mathcal{M} be the category of finitely generated graded left and right E -modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for homogeneous elements $a \in E$, $m \in M$. For example, if $J \subseteq E$ is a graded ideal, then E/J belongs to \mathcal{M} .

Following [AAH00] we call a linear form $v \in E_1$ *regular* on $M \in \mathcal{M}$ if the annihilator of v is the smallest possible, i.e., if $0 :_M v = v \cdot M$. A regular sequence on M is a sequence v_1, \dots, v_r in E_1 such that v_i is regular on $M/(v_1, \dots, v_{i-1})M$ for $1 \leq i \leq r$ and $M/(v_1, \dots, v_r)M \neq 0$. The *exterior depth* of M is also introduced in [AAH00] as the length of a maximal regular sequence and denoted by $\text{depth}_E M$. In the second section of this paper we study the relation between the depth of a squarefree S -module and the exterior depth of the corresponding squarefree E -module.

Using some results of Aramova, Avramov and Herzog [AAH00] and Römer [Röm01] we are able to show in Theorem 2.4 that for a squarefree S -module N and its corresponding squarefree E -module N_E it holds that

$$(1) \quad \text{depth}_E(N_E) \leq \text{depth}_S(N).$$

As byproduct we additionally obtain the inequalities

$$(2) \quad \text{cx}_E(N_E) \geq \text{proj dim}_S(N) \quad \text{and} \quad \text{depth}_S(N) - \text{depth}_E(N_E) \leq \text{reg}_S(N),$$

where $\text{cx}_E(N_E)$ is the complexity of N_E (see Section 2 for a definition).

It is an interesting question to classify all modules for which equality holds in the second inequality of (2). We show in Lemma 2.5 that this is the case for $K[\Delta]$ if Δ is a Cohen-Macaulay simplicial complex or if its Stanley-Reisner ideal has a linear resolution. But one easily finds examples of non-Cohen-Macaulay complexes for which equality holds even though their Stanley-Reisner ideal has no linear resolution (cf. Example 2.6).

Starting with a simplicial complex Δ one can construct its so-called exterior shifting Δ^e (see Section 2 for more details). This passage corresponds to the transition of an ideal in the exterior algebra to its generic initial ideal. It allows to compute some invariants defined in terms of an ideal and a generic basis by computing the corresponding invariants for the generic initial ideal and the standard basis. This idea goes back a long way, see e.g., [AH00], [AHH00], [BK88], [HT99], [Kal01].

The exterior depths of $K\{\Delta\}$ and $K\{\Delta^e\}$ coincide as was shown in [HT99, Proposition 2.3]. In Theorem 2.9 we describe the structure of the exterior shifted complex of Δ in terms of the exterior depth of $K\{\Delta\}$, i.e., we show that $\text{depth}_E(K\{\Delta\}) = r$ if and only if $\Delta^e = 2^{[r]} * \Gamma$. Here, Γ is a non-acyclic simplicial complex, $2^{[r]}$ denotes the $(r-1)$ -simplex.

Finally, we construct for every triple (s, t, r) of natural numbers with $r \geq s - t \geq 0$ a simplicial complex Δ such that $\text{depth}_E(K\{\Delta\}) = s$, $\text{depth}_S(K[\Delta]) = t$ and $\text{reg}_S(K[\Delta]) = r$.

In the third section our discussion is concentrated on the *exterior annihilator numbers* $\alpha_{i,j}(v_1, \dots, v_n; M)$ of an E -module M with respect to a sequence v_1, \dots, v_n of linear forms. These numbers are the exterior analogue of the symmetric annihilator numbers, first defined by Trung in [Tru87] and subsequently studied by Conca, Herzog and Hibi in [CHH04]. The results in this section are very much in the spirit of their results. We show that there is a non-empty Zariski-open set on which the annihilator numbers are constant. This gives rise to the definition of the *exterior generic annihilator numbers* $\alpha_{i,j}(E/J)$ for a graded ideal J in E . These numbers are a refinement of the exterior depth. For the exterior face ring of a simplicial complex they have a nice combinatorial interpretation, namely (see Corollary 3.10)

$$\alpha_{i,j}(K\{\Delta\}) = |\{F \in \Delta^e : |F| = j, [i] \cap F = \emptyset, F \cup \{i\} \notin \Delta^e\}|.$$

Using this combinatorial description we can express in Proposition 3.12 the symmetric Betti numbers of $K[\Delta^e]$ as a linear combination of certain exterior generic annihilator numbers, more precisely

$$\beta_{i,i+j}^S(K[\Delta^e]) = \sum_{l=1}^n \binom{n-l-j}{i-1} \alpha_{l,j}(K\{\Delta\}).$$

Finally, we show in Theorem 3.15 that the so-called Cartan-Betti numbers (see [NRV08, Definition 2.2]) can be bounded from above by a positive linear combination of the exterior generic annihilator numbers. In particular, this gives rise to an upper bound for the exterior Betti numbers,

$$\beta_{i,i+j}^E(E/J) \leq \sum_{k=1}^n \binom{n+i-k-1}{i-1} \alpha_{k,j}(E/J) \quad i \geq 1, j \geq 0,$$

which has been also obtained in another way in [NRV08, Theorem 2.4(i)]. Furthermore, this bound is tight if and only if J is componentwise linear.

In the fourth section we discuss the issue if the generic annihilator numbers are the minimal ones among all annihilator numbers with respect to a sequence. By minimal we mean that for any basis $\{v_1, \dots, v_n\}$ of E_1 , $1 \leq i \leq n$ and $j \in \mathbb{Z}$ it should hold that

$$\alpha_{i,j}(E/J) \leq \alpha_{i,j}(v_1, \dots, v_n; E/J).$$

We give an example, where this is not the case (see Example 4.2). After a slight modification the same example shows that also in the symmetric case this minimality is not given (see Example 4.5).

2. EXTERIOR DEPTH

Yanagawa introduced squarefree modules over the polynomial ring as a generalization of squarefree monomial ideals in [Yan00]. We fix some notations. For $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ we say that a is *squarefree* if $0 \leq a_i \leq 1$ for $i = 1, \dots, n$. We set $|a| = a_1 + \dots + a_n$ and $\text{supp}(a) = \{i : a_i \neq 0\}$.

A finitely generated \mathbb{N}^n -graded S -module $N = \bigoplus_{a \in \mathbb{N}^n} N_a$ is called *squarefree* if the multiplication map $N_a \rightarrow N_{a+\varepsilon_i}$, $y \mapsto x_i y$ is bijective for any $a \in \mathbb{N}^n$ and for all $i \in \text{supp}(a)$, where $\varepsilon_i \in \mathbb{N}^n$ is the vector with 1 at the i -th position and zero otherwise.

Römer defined in [Röm01, Definition 1.4] a finitely generated \mathbb{N}^n -graded E -module $M = \bigoplus_{a \in \mathbb{N}^n} M_a$ to be *squarefree* if it has only squarefree (non-zero) components.

Aramova, Avramov and Herzog and Römer construct in [AAH00] and [Röm01] a squarefree E -module N_E and its minimal free resolution starting from a squarefree S -module N and its minimal free resolution.

Theorem 2.1. [Röm01, Theorem 1.2] *The assignment $N \mapsto N_E$ induces an equivalence between the categories of squarefree S -modules and squarefree E -modules (where the morphisms are the \mathbb{N}^n -graded homomorphisms).*

For $F = \{i_1, \dots, i_r\} \subseteq [n]$ we set $x_F = x_{i_1} \cdots x_{i_r}$ and $e_F = e_{i_1} \wedge \dots \wedge e_{i_r}$. We usually assume that $1 \leq i_1 < \dots < i_r \leq n$. The elements e_F are called *monomials* in E .

Example 2.2. Let Δ be a simplicial complex on the vertex set $[n]$. Recall that a simplicial complex Δ on vertex set $[n]$ is a collection of subsets of $[n]$ such that whenever $F \in \Delta$ and $G \subseteq F$ it holds that $G \in \Delta$. The elements of Δ are called *faces*. Throughout this paper we always assume that $i \in \Delta$ for all $i \in [n]$.

Let $K[\Delta] = S/I_\Delta$ be the *Stanley-Reisner ring* of Δ , where I_Δ is the Stanley-Reisner ideal $I_\Delta = (x_F : F \notin \Delta)$ of Δ , and let $K\{\Delta\} = E/J_\Delta$ be the *exterior face ring*, where J_Δ is the exterior face ideal $J_\Delta = (e_F : F \notin \Delta)$ of Δ .

The Stanley-Reisner ring $K[\Delta]$ is a typical example of a squarefree S -module (see [Yan00]), the exterior face ring $K\{\Delta\}$ of a squarefree E -module (see [Röm01]).

These two correspond to each other under the equivalence of categories, i.e. $K[\Delta]_E \cong K\{\Delta\}$ (see [AAH00, Theorem 1.3]).

Our aim is to show that the symmetric depth of a squarefree S -module N is always greater or equal than the exterior depth of the corresponding squarefree E -module N_E . Recall from the introduction that the depth of an E -module $M \in \mathcal{M}$ is the length of a maximal M -regular sequence. It is strongly related with the *complexity* $\text{cx}_E(M)$ of the module which is defined as

$$\text{cx}_E(M) = \inf\{c \in \mathbb{Z} : \beta_i^E(M) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and for all } i \geq 1\}.$$

In other words, the complexity measures the polynomial growth of the exterior Betti numbers of M and is therefore a measure for the size of a minimal free resolution of M by free E -modules.

The proof is based on comparisons of several invariants of modules over S or E . We collect the formulas used for these comparisons in the next theorem.

Theorem 2.3. (i) [Röm01, Corollary 1.3] *Let N be a squarefree S -module and let N_E be the associated squarefree E -module. Let $\beta_{i,j}^E$ and $\beta_{i,j}^S$ denote the graded Betti numbers over E and S , respectively. Then*

$$\beta_{i,i+j}^E(N_E) = \sum_{k=0}^i \binom{i+j-1}{j+k-1} \beta_{k,k+j}^S(N).$$

(ii) (e.g., [BH98, Theorem 1.3.3]) *Let N be a finitely generated S -module. Then*

$$\text{proj dim}_S(N) + \text{depth}_S(N) = n.$$

(iii) [AAH00, Theorem 3.2] *Let $M \in \mathcal{M}$. Then*

$$\text{depth}_E(M) + \text{cx}_E(M) = n.$$

We have now provided all notions and facts we need to prove the desired inequality.

Theorem 2.4. *Let N be a finitely generated \mathbb{N}^n -graded squarefree S -module. Then*

$$0 \leq \text{depth}_S(N) - \text{depth}_E(N_E) = \text{cx}_E(N_E) - \text{proj dim}_S(N) \leq \text{reg}_S(N).$$

Proof. A comparison of Theorem 2.3 (ii) and (iii) shows that the differences between the depths and between the complexity and the projective dimension are equal. We show that the inequalities are satisfied by the latter difference.

From Theorem 2.3(i) it follows that

$$\beta_i^E(N_E) = \sum_{j \geq 0} \beta_{i,i+j}^E(N_E) = \sum_{j \geq 0} \sum_{k=0}^i \binom{i+j-1}{j+k-1} \beta_{k,k+j}^S(N).$$

Let $m_j^{(i)} = \max\{k+j : \beta_{k,k+j}^S(N) \neq 0, 0 \leq k \leq i\}$. From the above formula for the Betti numbers we conclude that $\beta_{i,i+j}^E(N_E)$ is a polynomial in i of degree $m_j^{(i)} - 1$. Therefore, it follows that $\beta_i^E(N_E)$ is a polynomial in i of degree $m^{(i)} - 1$, where $m^{(i)} = \max\{m_j^{(i)} : j \geq$

$0, \beta_{i,i+j}^E \neq 0\}$ (here we can write max instead of sup because the modules in the minimal free E -resolution of a finitely generated module are finitely generated). This yields for the complexity

$$\begin{aligned} \text{cx}_E(N_E) &= \sup\{m^{(i)} : i \geq 0\} \\ &= \sup\{k+j : \beta_{k,k+j}^S(N) \neq 0, k \geq 0, j \geq 0\} \\ &= \max\{l : \beta_{i,l}^S \neq 0, i \geq 0, l \geq 0\}, \end{aligned}$$

where the last equality holds because N has a finite S -resolution. Let $p = \text{proj dim}_S(N)$. Then there exists $0 \leq k \leq \text{reg}_S(N)$ such that $\beta_{p,p+k}^S(N) \neq 0$. This implies $\text{cx}_E(N_E) \geq p+k \geq p = \text{proj dim}_S(N)$. This finally shows the claim. \square

An interesting question to ask is if there are classes of squarefree modules for which equality holds in the second inequality in Theorem 2.4. In the special case of Stanley-Reisner rings of simplicial complexes we can identify at least two such classes.

Lemma 2.5. *Let Δ be a simplicial complex. If J_Δ has a linear resolution or if Δ is Cohen-Macaulay, then*

$$\text{depth}_S(K[\Delta]) - \text{depth}_E(K\{\Delta\}) = \text{cx}_E(K\{\Delta\}) - \text{proj dim}_S(K[\Delta]) = \text{reg}_S(K[\Delta]).$$

Proof. Examining the proof of Theorem 2.4 we see that we have an equality if and only if $\beta_{p,p+r}^S(K[\Delta]) \neq 0$ where $p = \text{proj dim}_S(K[\Delta])$ and $r = \text{reg}_S(K[\Delta])$. This is the most right lower corner in the Betti diagram. It is obviously the case if I_Δ has a linear resolution.

If Δ is Cohen-Macaulay, $K[\Delta]$ has depth d over S where $\dim \Delta = d - 1$ and the face ring $K\{\Delta\}$ has a d -linear injective resolution over E (see e.g., [AH00, Corollary 7.6] or [KR09, Example 5.1]). Following [KR09, Theorem 5.3] it holds that

$$d = \text{depth}_E(K\{\Delta\}) + \text{reg}_E(K\{\Delta\}).$$

The regularity over E is the same as the regularity over S (this follows from the relation between the Betti numbers, Theorem 2.3(i)), hence

$$\text{reg}_S(K[\Delta]) = d - \text{depth}_E(K\{\Delta\}) = \text{depth}_S(K[\Delta]) - \text{depth}_E(K\{\Delta\}).$$

As an alternative proof of this case it is possible to show that if Δ is Cohen-Macaulay then $K[\Delta]$ has only one extremal Betti number. \square

The following example shows that in general the converse of Lemma 2.5 is not true.

Example 2.6. Let Δ be the simplicial complex on the vertex set $[4]$ consisting of two triangles – one of them filled, the other one missing – glued together along one edge. The face ideal is $J_\Delta = (e_3e_4, e_1e_2e_4)$. It is not generated in one degree and thus does not have a linear resolution. On the other hand Δ is not pure so it cannot be Cohen-Macaulay. But I_Δ is squarefree stable whence we can compute the depths and the regularity from known formulas (A monomial ideal I in S is called *squarefree stable*, if for all squarefree monomials $x_A \in S$ and all $i > \min(A)$ with $i \notin A$ one has $x_i x_{A \setminus \min(A)} \in I$, see Example 2.11 for the formulas)). Then

$$\begin{aligned} \text{depth}_S(K[\Delta]) &= \min\{\min(u) + \deg(u) : u \in G(I_\Delta)\} - 2 = 2, \\ \text{reg}_S(K[\Delta]) &= \max\{\deg(u) : u \in G(I_\Delta)\} - 1 = 2, \end{aligned}$$

$$\text{depth}_E(K\{\Delta\}) = \min\{\min(u) : u \in G(J_\Delta)\} - 1 = 0.$$

In the remainder of the section we continue to discuss the case of simplicial complexes. Starting with a simplicial complex Δ one can construct its so-called exterior shifting (see [Kal01] for more details). A simplicial complex Δ on a ground set $[n]$ is *shifted* if for every face $F \in \Delta$, $i \in F$ and $j < i$ we have that $(F \setminus \{i\}) \cup \{j\} \in \Delta$. (Algebraists would probably prefer $j > i$ here.) Let now Δ be a simplicial complex on vertex set $[n]$. Let K'/K be a field extension containing the algebraically independent elements a_{ij} , $1 \leq i, j \leq n$ over K , and let f_1, \dots, f_n with $f_i = \sum_{j=1}^n a_{ij}e_j$ for $1 \leq i \leq n$ be a generic basis of E'_1 where $E' = K' \otimes_K E$. For $A = \{i_1 < \dots < i_r\} \subseteq [n]$ we set $f_A = f_{i_1} \wedge \dots \wedge f_{i_r} \in E$ and by \bar{f}_A we denote the image of f_A in $K'\{\Delta\}$. Let further $<_{\text{lex}}$ denote the lexicographic order on subsets of \mathbb{N} of the same size, i.e., $A <_{\text{lex}} B$ if and only if $\min((A \setminus B) \cup (B \setminus A)) \in A$. In order to define the exterior shifting of Δ we define the shifting of a family of sets of equal cardinality. We set

$$\Delta_i^e = \left\{ A \in \binom{[n]}{i} : \bar{f}_A \notin \text{span}\{\bar{f}_{A'} \mid A' <_{\text{lex}} A\} \right\}$$

for $0 \leq i \leq \dim \Delta + 1$. Finally, the exterior shifting of Δ is the following simplicial complex $\Delta^e = \bigcup_{i=0}^{\dim \Delta + 1} \Delta_i^e$. Observe that Δ^e may depend on K , but not on K' . One can show that Δ^e is indeed a shifted simplicial complex having the same f -vector as Δ . Using the definition of the generic initial ideal of an ideal $J \subseteq E$ one can easily show that $J_{\Delta^e} = \text{gin}_{<_{\text{rlex}}}(J_\Delta)$. Here $<_{\text{rlex}}$ denotes the reverse lexicographic order with respect to $e_1 < \dots < e_n$. We therefore can compute the exterior shifting of a simplicial complex by computing the generic initial ideal of the exterior face ideal J_Δ .

The following remark shows that the exterior depth is unchanged by exterior shifting.

Remark 2.7. [HT99, Proposition 2.3] Let $J \subseteq E$ be a graded ideal. Then

$$\text{depth}_E(E/J) = \text{depth}_E(E/\text{gin}_{<_{\text{rlex}}}(J)),$$

where $\text{gin}_{<_{\text{rlex}}}(J)$ denotes the generic initial ideal of J with respect to $<_{\text{rlex}}$.

A monomial ideal J in E is called *stable*, if for all monomials $e_A \in J$ and all $i > \min(A)$ with $i \notin A$ one has $e_i e_{A \setminus \{\min(A)\}} \in J$. It is called *strongly stable*, if for all monomials $e_A \in J$ and all $i > j$ with $j \in A$, $i \notin A$ one has $e_i e_{A \setminus \{j\}} \in J$. The face ideal of a simplicial complex is a strongly stable ideal if and only if the complex is shifted. Therefore the generic initial ideal is strongly stable (see also [AHH97, Proposition 1.7] for a direct proof). It is well-known that for stable ideals an initial segment of e_1, \dots, e_n up to the depth is a regular sequence. However, for the convenience of the reader we include a proof of it.

Lemma 2.8. *Let $J \subseteq E$ be a stable ideal of depth t . Then e_1, \dots, e_t is a regular sequence on E/J .*

Proof. If $t = 0$ there is nothing to prove, thus we assume $t > 0$. We show that e_1 is regular on $\text{gin}_{<_{\text{rlex}}}(J) = J$. Then the claim follows by induction on t (note that since J is stable also $J + (e_1)/(e_1)$ is stable in $E/(e_1) \cong K\langle e_2, \dots, e_n \rangle$).

Being a regular sequence is an open condition since it is equivalent to the vanishing of the first Cartan homology with respect to the sequence (see [AAH00, Remark 3.4] for the result and e.g., Section 3 for a definition of Cartan homology). Hence there exists an

E/J -regular sequence v_1, \dots, v_t of linear forms such that there is a generic automorphism g mapping e_i to v_i and $\text{gin}(J) = \text{in}(g(J))$. Since the considered monomial order is the revlex order, we have $\text{in}(g(J) + e_1) = \text{in}(g(J)) + (e_1)$ and $\text{in}(g(J) : e_1) = \text{in}(g(J)) : e_1$ by [AH00, Proposition 5.1] (using the reversed order on $[n]$). Then the Hilbert functions

$$\begin{aligned} H(-, E/\text{gin}(J + v_1)) &= H(-, E/(J + (v_1))) = H(-, E/g(J + (v_1))) \\ &= H(-, E/(g(J) + e_1)) = H(-, E/\text{in}(g(J) + (e_1))) = H(-, E/(\text{gin}(J) + (e_1))) \end{aligned}$$

are equal. Analogously one sees $H(-, E/\text{gin}(J : v_1)) = H(-, E/\text{gin}(J) : e_1)$. On the other hand $J + (v_1) = J : v_1$ as v_1 is E/J -regular, hence $H(-, E/\text{gin}(J + (v_1))) = H(-, E/\text{gin}(J : v_1))$. Thus the Hilbert functions of $\text{gin}(J) + (e_1)$ and $\text{gin}(J) : e_1$ coincide as well which already implies $\text{gin}(J) + (e_1) = \text{gin}(J) : e_1$ because $\text{gin}(J) + (e_1) \subseteq \text{gin}(J) : e_1$ is clear. This means that e_1 is $E/\text{gin}(J)$ -regular. \square

The next theorem characterizes simplicial complexes with a certain exterior depth in terms of the exterior shifting. For a linear form $v \in E_1$ and $M \in \mathcal{M}$ there is the complex

$$(M, v) : \quad \dots \longrightarrow M_{j-1} \xrightarrow{\cdot v} M_j \xrightarrow{\cdot v} M_{j+1} \longrightarrow \dots,$$

since $v^2 = 0$. Then v is M -regular if and only if all homology modules $H^j(M, v)$ of the complex vanish for all $j \in \mathbb{Z}$.

If $v \in E_1$ is a generic element then following [AH00, Lemma 3.3] the homology of the complex $(K\{\Delta\}, v)$ is isomorphic to the reduced simplicial homology of Δ (see e.g., [BH98, Section 5.3] for a definition of reduced simplicial homology). Therefore, we have $\text{depth}_E(K\{\Delta\}) = 0$ if and only if there exists $0 \leq i \leq \dim \Delta$ such that $\tilde{H}_i(\Delta; K) \neq 0$. A simplicial complex Δ with this property is called *non-acyclic*.

Theorem 2.9. *Let Δ be a simplicial complex on vertex set $[n]$. Then $\text{depth}_E(K\{\Delta\}) = r$ if and only if $\Delta^e = 2^{[r]} * \Gamma$, where Γ is a non-acyclic simplicial complex, $2^{[r]}$ the $(r-1)$ -simplex and $\dim \Gamma = \dim \Delta - r$.*

Proof. We first assume that $\text{depth}_E(K\{\Delta\}) = r$. In order to prove the statement we need to show that for $F \in \Delta^e$ it holds that $F \cup [t]$ is a face of Δ^e for any $t \leq r$. Let $F \in \Delta^e$. Without loss of generality we may assume that $F \cap [t] = \emptyset$. We can take e_1, \dots, e_t as a regular sequence for the exterior face ring $K\{\Delta^e\} = E/\text{gin}_{<\text{rlex}}(J_\Delta)$ by Lemma 2.8. Suppose $F \in \Delta^e$ such that $F \cap [t] = \emptyset$ and $F \cup [t] \notin \Delta^e$. Thus, $e_t \wedge \dots \wedge e_1 \wedge e_F = 0$ in $K\{\Delta^e\}$. By the definition of a regular sequence it follows that

$$e_{t-1} \wedge \dots \wedge e_1 \wedge e_F \in (e_t) \subseteq K\{\Delta^e\}.$$

Therefore, $e_{t-1} \wedge \dots \wedge e_1 \wedge e_F = 0 \in E/(\text{gin}_{<\text{rlex}}(J_\Delta) + (e_t))$. Inductively, we get $e_F = 0 \in E/(\text{gin}_{<\text{rlex}}(J_\Delta) + (e_t, \dots, e_1))$. Since $F \cap [t] = \emptyset$ by assumption, we have $e_F \in \text{gin}_{<\text{rlex}}(J_\Delta)$, i.e., $F \notin \Delta^e$. This is a contradiction.

This shows $\Delta^e = 2^{[r]} * \Gamma$ for some $(\dim \Delta - r)$ -dimensional simplicial complex Γ with $J_\Gamma = \text{gin}_{<\text{rlex}}(J_\Delta) + (e_1, \dots, e_r)$. By definition of the exterior depth and Lemma 2.8 it holds that $\text{depth}_E(K\{\Gamma\}) = \text{depth}_E(K\{\Delta^e\}) - r = 0$. We therefore conclude that $\tilde{H}_i(\Gamma; K) \neq 0$ for some $0 \leq i \leq \dim(\Gamma)$, i.e., Γ is non-acyclic.

In order to prove the sufficiency part recall that Remark 2.7 implies that $\text{depth}_E(K\{\Delta\}) = \text{depth}_E(K\{\Delta^e\})$. Thus, we only need to show that $\text{depth}_E(K\{\Delta^e\}) = r$. By assumption,

we have that $\Delta^e = 2^{[r]} * \Gamma$, where Γ is a non-acyclic simplicial complex. Then the sequence e_1, \dots, e_r is regular on $K\{\Delta^e\}$ by Lemma 2.8. This implies $\text{depth}_E(K\{\Delta^e\}) \geq r$. It further holds that $K\{\Delta^e\}/(e_1, \dots, e_r) \cong K\{\Gamma\}$. Since Γ is non-acyclic we know from the remarks preceding this theorem that $\text{depth}_E(K\{\Gamma\}) = 0$, i.e., there does not exist any regular element on $K\{\Gamma\}$. Using that each regular sequence on $K\{\Delta^e\}$ can be extended to a maximal one, we therefore deduce that e_1, \dots, e_r is already maximal and thus it follows that $\text{depth}_E(K\{\Delta^e\}) = r$. \square

Remark 2.10. The above theorem can be used to deduce two of the previous results in the special case of simplicial complexes.

- (i) Note that from the above proof the inequality

$$\text{depth}_E(K\{\Delta\}) \leq \text{depth}_S(K[\Delta])$$

between the exterior and the symmetric depth can be deduced. If $\text{depth}_E(K\{\Delta\}) = r$, then Theorem 2.9 implies $\Delta^e = 2^{[r]} * \Gamma$ and therefore x_1, \dots, x_r is a regular sequence for $K[\Delta^e]$, i.e., $\text{depth}_S(K[\Delta^e]) \geq r$. Since $\text{depth}_S(K[\Delta]) = \text{depth}_S(K[\Delta^e])$ (this follows from [AH00, Theorem 9.7] which states that the extremal Betti numbers of $K[\Delta]$ and $K[\Delta^e]$ coincide whence the projective dimensions and thus the depths by Auslander-Buchsbaum (cf. Theorem 2.3(ii)) of both are equal) we obtain the required inequality.

- (ii) Using the characterization of Theorem 2.9 we can give a second proof of Lemma 2.5. If Δ is non-acyclic, i.e., $\tilde{H}_i(\Delta; K) \neq 0$ for some $0 \leq i \leq \dim \Delta$, we have $\text{depth}_E(K\{\Delta\}) = 0$. Since Δ is Cohen-Macaulay it follows from Reisner's criterion that $\tilde{H}_i(\Delta; K) = 0$ for $i < \dim \Delta$. Thus, $\tilde{H}_{\dim \Delta}(\Delta; K) \neq 0$. From [KW08, Proposition 2.6] we conclude that $\text{reg}_S(K[\Delta]) = \dim \Delta + 1$. Combining these two facts and using that Δ is Cohen-Macaulay, i.e., $\text{depth}_S(K[\Delta]) = \dim_S(K[\Delta])$, we obtain

$$\begin{aligned} \text{depth}_S(K[\Delta]) - \text{depth}_E(K\{\Delta\}) &= \text{depth}_S(K[\Delta]) \\ &= \dim_S K[\Delta] = \dim \Delta + 1 = \text{reg}_S(K[\Delta]). \end{aligned}$$

Let us now assume that Δ is an acyclic simplicial complex and let $\text{depth}_E(K\{\Delta\}) = r$. We may assume that $\Delta = \Delta^e$ since Δ and Δ^e have the same symmetric and exterior depth, respectively, and the same regularity (see [Röm01, Corollary 1.3]). From Theorem 2.9 we know that $\Delta^e = 2^{[r]} * \Gamma$, where Γ is a non-acyclic simplicial complex. In particular, since Δ^e is Cohen-Macaulay, so is Γ . This implies

$$\text{depth}_S(K[\Gamma]) = \dim_S(K[\Gamma]) = \dim_S(K[\Delta]) - r = \text{depth}_S(K[\Delta]) - r.$$

Using that $\text{depth}_E(K\{\Gamma\}) = 0 = \text{depth}_E(K\{\Delta\}) - r$ we deduce

$$\text{depth}_S(K[\Delta]) - \text{depth}_E(K\{\Delta\}) = \text{depth}_S(K[\Gamma]) - \text{depth}_E(K\{\Gamma\}).$$

Since Γ is a non-acyclic simplicial complex we know from the first part of our considerations that $\text{depth}_S(K[\Gamma]) - \text{depth}_E(K\{\Gamma\}) = \text{reg}_S(K[\Gamma])$. We further have that $K[\Gamma] \cong K[\Delta]/(x_1, \dots, x_r)$ and x_1, \dots, x_r is a regular sequence on $K[\Delta]$. Since reducing modulo a regular sequence leaves the regularity unchanged (see e.g.,

[Eis04, Proposition 20.20] it holds that $\text{reg}_S(K[\Delta^e]) = \text{reg}_S(K[\Gamma])$. This finally shows the claim.

A natural question to ask is which triples of numbers (t, s, r) with $r \geq s - t \geq 0$ can occur such that there exists a simplicial complex Δ with the property that $\text{depth}_E(K\{\Delta\}) = t$, $\text{depth}_S(K[\Delta]) = s$ and $\text{reg}_S(K[\Delta]) = r$. We can answer this issue by showing that all triples of numbers are possible.

For a monomial ideal I in the polynomial ring or in the exterior algebra we denote by $G(I)$ the unique minimal monomial generating set of I . For a monomial $u \in S$ let $\min(u) = \min\{i : x_i \text{ divides } u\}$ and analogously $\min(u) = \min\{i : e_i \text{ divides } u\}$ for a monomial $u \in E$. Recall that a monomial ideal $I \subseteq S$ is called *squarefree stable* if for all squarefree monomials $u \in I$ and all $i > \min(u)$ such that x_i does not divide u one has $x_i \left(\frac{u}{x_{\min(u)}} \right) \in I$.

Example 2.11. Let s, t, r be natural numbers with $r \geq s - t \geq 0$. We construct a simplicial complex Δ with the property that $\text{depth}_E(K\{\Delta\}) = t$, $\text{depth}_S(K[\Delta]) = s$ and $\text{reg}_S(K[\Delta]) = r$ as follows (note that if $r = 0$ there must be $i \in [n]$, $\{i\} \notin \Delta$, violating our assumption made on simplicial complexes in the rest of the paper).

Let $n = t + r + 3$ and Δ be the simplicial complex on the ground set $[n]$ with minimal non-faces

- $\{n, n-1, \dots, n-(s-t)+1, i\}$ with $i = n-(s-t), \dots, t+1$,
- $\{n-r-1, n-r, \dots, n-2, n-1\}$,
- $\{n-r-1, n-r, \dots, \hat{j}, \dots, n-1, n\}$ with $j = n-(s-t)+1, \dots, n-1$,

where \hat{j} means that j is omitted. By construction J_Δ is a stable and I_Δ a squarefree stable ideal. There are the following formulas to compute the depth and the regularity of such ideals (taking into account the reverse order on $[n]$).

- $\text{depth}_S(K[\Delta]) = \min\{\min(u) + \deg(u) : u \in G(I_\Delta)\} - 2$ ([AHH98, Corollary 2.4]);
- $\text{reg}_S(K[\Delta]) = \max\{\deg(u) : u \in G(I_\Delta)\} - 1$ ([AHH98, Corollary 2.6]);
- $\text{depth}_E(K\{\Delta\}) = \min\{\min(u) : u \in G(J_\Delta)\} - 1$ ([KR09, Proposition 3.4]).

We compute these invariants:

$$\begin{aligned} \text{depth}_S(K[\Delta]) &= \min\{t+1+(s-t+1), n-r-1+(r+1)\} - 2 \\ &= \min\{s+2, n\} - 2 \\ &= s \end{aligned}$$

because $n = t + r + 3 \geq s + 3$.

$$\text{reg}_S(K[\Delta]) = \max\{s-t+1, r+1\} - 1 = r$$

because $r \geq s - t$.

$$\text{depth}_E(K\{\Delta\}) = \min\{t+1, n-r-1\} - 1 = t$$

because $n - r - 1 = t + 2$.

3. EXTERIOR GENERIC ANNIHILATOR NUMBERS

In this section we introduce the so-called exterior generic annihilator numbers of an E -module M which are the exterior analogue of the symmetric generic annihilator numbers of an S -module N introduced first by Trung in [Tru87] and studied later by Conca, Herzog and Hibi (see [CHH04]). In what follows we derive some facts which relate those numbers to the symmetric Betti numbers of a module. In order to do so, we need to lay some background and introduce some notation.

We first recall the construction of the Cartan complex from [AHH97, Section 2]. It is a very useful complex over E which plays a similar role as the Koszul complex for the polynomial ring. For a sequence $\mathbf{v} = v_1, \dots, v_m$ with $v_i \in E_1$ let $C_\bullet(\mathbf{v}; E) = C_\bullet(v_1, \dots, v_m; E)$ be the free divided power algebra $E\langle x_1, \dots, x_m \rangle$. It is generated by the divided powers $x_i^{(j)}$ for $1 \leq i \leq m$ and $j \geq 0$ which satisfy the relations $x_i^{(j)} x_i^{(k)} = ((j+k)!/(j!k!)) x_i^{(j+k)}$. Thus $C_i(\mathbf{v}; E)$ is a free E -module with basis $x^{(a)} = x_1^{(a_1)} \cdots x_m^{(a_m)}$, $a \in \mathbb{N}^m$, $|a| = i$. The E -linear differential on $C_\bullet(v_1, \dots, v_m; E)$ is

$$\begin{aligned} \partial_i : C_i(v_1, \dots, v_m; E) &\longrightarrow C_{i-1}(v_1, \dots, v_m; E) \\ x^{(a)} &\longmapsto \sum_{a_j > 0} v_j x_1^{(a_1)} \cdots x_j^{(a_j-1)} \cdots x_m^{(a_m)}. \end{aligned}$$

One easily sees that $\partial \circ \partial = 0$. So this is indeed a complex.

Definition 3.1. Let $M \in \mathcal{M}$. The complex

$$C_\bullet(\mathbf{v}; M) = C_\bullet(\mathbf{v}; E) \otimes_E M$$

is called the *Cartan complex* of \mathbf{v} with values in M . The corresponding homology modules

$$H_i(\mathbf{v}; M) = H_i(C_\bullet(\mathbf{v}; M))$$

are called the *Cartan homology* of \mathbf{v} with values in M .

Cartan homology can be computed inductively as there is a long exact sequence connecting the homologies of v_1, \dots, v_j and v_1, \dots, v_j, v_{j+1} for $j = 1, \dots, m-1$.

To begin with there exists an exact sequence of complexes

$$0 \longrightarrow C_\bullet(v_1, \dots, v_j; M) \xrightarrow{\iota} C_\bullet(v_1, \dots, v_{j+1}; M) \xrightarrow{\tau} C_\bullet(v_1, \dots, v_{j+1}; M)(-1) \longrightarrow 0,$$

where (-1) indicates a shift in the homological degree, ι is the natural inclusion map and τ is given by

$$\tau(g_0 + g_1 x_{j+1} + \dots + g_k x_{j+1}^{(k)}) = g_1 + g_2 x_{j+1} + \dots + g_k x_{j+1}^{(k-1)},$$

where the g_i belong to $C_{k-i}(v_1, \dots, v_j; M)$.

This exact sequence induces a long exact sequence of homology modules.

Proposition 3.2. [AH00, Propositions 4.1] *Let $M \in \mathcal{M}$ and $\mathbf{v} = v_1, \dots, v_m \in E_1$. For all $1 \leq j \leq m$ there exists a long exact sequence of graded E -modules*

$$\begin{aligned} \dots \longrightarrow H_i(v_1, \dots, v_j; M) &\xrightarrow{\alpha_i} H_i(v_1, \dots, v_{j+1}; M) \xrightarrow{\beta_i} H_{i-1}(v_1, \dots, v_{j+1}; M)(-1) \\ &\xrightarrow{\delta_{i-1}} H_{i-1}(v_1, \dots, v_j; M) \longrightarrow H_{i-1}(v_1, \dots, v_{j+1}; M) \longrightarrow \dots \end{aligned}$$

Here α_i is induced by the inclusion map ι , β_i by τ and δ_{i-1} is the connecting homomorphism, which acts as follows: if $z = g_0 + g_1x_{j+1} + \dots + g_{i-1}x_{j+1}^{(i-1)}$ is a cycle in $C_{i-1}(v_1, \dots, v_{j+1}; M)$, then $\delta_{i-1}(\bar{z}) = \overline{g_0v_{j+1}}$.

Setting $\deg x_i = 1$ induces a grading on the complex and its homologies.

The Cartan complex $C_\bullet(v_1, \dots, v_m; E)$ with values in E is exact if the linear forms are K -linearly independent (see e.g., [AAH00, Remark 3.4(3)] or the proof of [AHH97, Theorem 2.2]) and hence it is a minimal graded free resolution of $H_0(v_1, \dots, v_m; E) = E/(v_1, \dots, v_m)$ over E . Thus it can be used to compute $\text{Tor}_i^E(E/(v_1, \dots, v_m), \cdot)$.

Proposition 3.3. [AHH97, Theorem 2.2] *Let $M \in \mathcal{M}$ and $\mathbf{v} = v_1, \dots, v_m \in E_1$ linearly independent over K . There are isomorphisms of graded E -modules*

$$\text{Tor}_i^E(E/(v_1, \dots, v_m), M) \cong H_i(\mathbf{v}; M) \quad \text{for all } i \geq 0.$$

Recall that $H^j(M, v)$ for $j \in \mathbb{Z}$ denotes the j -th homology of the complex

$$(3) \quad (M, v) : \quad \dots \longrightarrow M_{j-1} \xrightarrow{\cdot v} M_j \xrightarrow{\cdot v} M_{j+1} \longrightarrow \dots$$

and that v is M -regular if and only if $H^j(M, v) = 0$ for all $j \in \mathbb{Z}$.

We have now laid the required background in order to give the definition of the exterior annihilator numbers with respect to a certain sequence.

Definition 3.4. Let v_1, \dots, v_n be a basis of E_1 and let $M \in \mathcal{M}$. The numbers

$$\alpha_{i,j}(v_1, \dots, v_n; M) = \dim_K H^j(M/(v_1, \dots, v_{i-1})M, v_i)$$

for $j \in \mathbb{Z}$ and $1 \leq i \leq n$ are called the *exterior annihilator numbers* of M with respect to v_1, \dots, v_n .

So far, we have defined exterior annihilator numbers for an E -module which do depend on the chosen sequence. The following theorem justifies the definition of the so-called exterior generic annihilator numbers which are independent of the sequence.

Theorem 3.5. *Let $J \subseteq E$ be a graded ideal. Then there exists a non-empty Zariski-open set $U \subseteq \text{GL}_n(K)$ such that*

$$\alpha_{i,j}(\gamma(e_1, \dots, e_n); E/J) = \alpha_{i,j}(e_1, \dots, e_n; E/\text{gin}_{<\text{rlex}}(J))$$

for all $\gamma = (\gamma_{i,j})_{1 \leq i, j \leq n} \in U$, where $\gamma(e_1, \dots, e_n) = (\gamma_{1,1}e_1 + \dots + \gamma_{n,1}e_n, \dots, \gamma_{1,n}e_1 + \dots + \gamma_{n,n}e_n)$ and $<\text{rlex}$ denotes the reverse lexicographic order with respect to $e_1 < \dots < e_n$.

Proof. Let $U' = \{\varphi \in \text{GL}_n(K) : \text{in}_{<\text{rlex}}(\varphi(J)) = \text{gin}_{<\text{rlex}}(J)\}$ be the non-empty Zariski-open set of linear transformations that can be used to compute the generic initial ideal of J . Set $U = \{\varphi^{-1} : \varphi \in U'\}$. Let $\gamma = \varphi^{-1} \in U$ and set $v_i = \gamma(e_i)$ for $1 \leq i \leq n$, i.e., $\varphi(v_i) = e_i$. As φ is an automorphism, $E/(J + (v_1, \dots, v_i))$ and $E/(\varphi(J) + (e_1, \dots, e_i))$ have the same Hilbert function. [AH00, Proposition 5.1] implies that $\text{in}_{<\text{rlex}}(\varphi(J) + (e_1, \dots, e_i)) = \text{gin}_{<\text{rlex}}(J) + (e_1, \dots, e_i)$ (observe that we use the reversed order on $[n]$). Therefore also $E/(J + (v_1, \dots, v_i))$ and $E/(\text{gin}_{<\text{rlex}}(J) + (e_1, \dots, e_i))$ have the same Hilbert function. The sequences

$$0 \longrightarrow H^j(E/(J + (v_1, \dots, v_{i-1})), v_i) \longrightarrow (E/(J + (v_1, \dots, v_i)))_j$$

$$\xrightarrow{\cdot v_i} (E/(J + (v_1, \dots, v_{i-1})))_{j+1} \longrightarrow (E/(J + (v_1, \dots, v_i)))_{j+1} \longrightarrow 0$$

and

$$\begin{aligned} 0 \longrightarrow H^j(E/(\operatorname{gin}_{<\operatorname{rlex}}(J) + (e_1, \dots, e_{i-1})), e_i) &\longrightarrow (E/(\operatorname{gin}_{<\operatorname{rlex}}(J) + (e_1, \dots, e_i)))_j \\ \xrightarrow{\cdot e_i} (E/(\operatorname{gin}_{<\operatorname{rlex}}(J) + (e_1, \dots, e_{i-1})))_{j+1} &\longrightarrow (E/(\operatorname{gin}_{<\operatorname{rlex}}(J) + (e_1, \dots, e_i)))_{j+1} \longrightarrow 0 \end{aligned}$$

are exact sequences of K -vector spaces (recall the definition of $H^j(-, -)$ from (3)). The vector space dimensions of the three latter vector spaces in the two sequences coincide, hence it holds that

$$\begin{aligned} \alpha_{i,j}(v_1, \dots, v_n; E/J) &= \dim_K H^j(E/(J + (v_1, \dots, v_{i-1})), v_i) \\ &= \dim_K H^j(E/(\operatorname{gin}_{<\operatorname{rlex}}(J) + (e_1, \dots, e_{i-1})), e_i) \\ &= \alpha_{i,j}(e_1, \dots, e_n; E/\operatorname{gin}_{<\operatorname{rlex}}(J)). \end{aligned}$$

□

As mentioned beforehand, now the following definition makes sense.

Definition 3.6. Let $J \subseteq E$ be a graded ideal. We set

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(e_1, \dots, e_n; E/\operatorname{gin}_{<\operatorname{rlex}}(J))$$

for $j \in \mathbb{Z}$ and $1 \leq i \leq n$ and call these numbers the *exterior generic annihilator numbers* of E/J .

Remark 3.7. Let $J \subseteq E$ be a graded ideal. By definition of the $\alpha_{i,j}$ and the fact that $\operatorname{gin}_{<\operatorname{rlex}}(\operatorname{gin}_{<\operatorname{rlex}}(J)) = \operatorname{gin}_{<\operatorname{rlex}}(J)$, it holds that

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(E/\operatorname{gin}_{<\operatorname{rlex}}(J)).$$

Set $\alpha_i(E/J) = \sum_{j \in \mathbb{Z}} \alpha_{i,j}(E/J)$ and $1 \leq r \leq n$. Then $\alpha_i(E/J) = 0$ for all $i \leq r$ if and only if $r \leq \operatorname{depth}_E(E/J)$. This is an easy consequence of the fact that e_1, \dots, e_i is a regular sequence on $E/\operatorname{gin}_{<\operatorname{rlex}}(J)$ if and only if $i \leq \operatorname{depth}_E(E/\operatorname{gin}_{<\operatorname{rlex}}(J)) = \operatorname{depth}_E(E/J)$ (see Lemma 2.8).

It is well-known that being a regular sequence is a Zariski-open condition. One can prove it directly using that a sequence of linear forms is regular if and only if the first Cartan homology vanishes. But using the generic annihilator numbers provides a short proof.

Proposition 3.8. Let $J \subseteq E$ be a graded ideal and $\operatorname{depth}_E E/J = t$. Then there exists a non-empty Zariski-open set $U \subseteq GL_n(K)$ such that $\gamma_{1,1}e_1 + \dots + \gamma_{n,1}e_n, \dots, \gamma_{1,t}e_1 + \dots + \gamma_{n,t}e_n$ is an E/J -regular sequence for all $\gamma = (\gamma_{i,j})_{1 \leq i,j \leq n} \in U$.

Proof. Let U be the non-empty Zariski-open set as in Proposition 3.5, i.e., such that the annihilator numbers with respect to sequences v_1, \dots, v_n induced by U equal the generic annihilator numbers. Following Lemma 2.8 e_1, \dots, e_t is a regular sequence on $E/\operatorname{gin}_{<\operatorname{rlex}}(J)$ and therefore

$$\alpha_{i,j}(v_1, \dots, v_n; E/J) = \alpha_{i,j}(e_1, \dots, e_n; E/\operatorname{gin}_{<\operatorname{rlex}}(J)) = 0$$

for $i \leq t$. Thus v_1, \dots, v_t is regular on E/J .

□

It is also well-known that the same statement is true over the polynomial ring. Nevertheless we were not able to give a reference for this fact. In [Swa06] Swartz gives a proof for the special case of Stanley-Reisner rings of simplicial complexes. Therefore we include a short proof following ideas from Herzog using almost regular sequences in Section 4 where these are defined.

We can describe the numbers $\alpha_{i,j}$ as follows.

Theorem 3.9. *Let $J \subseteq E$ be a graded ideal. Then*

$$\alpha_{i,j}(E/J) = |\{\overline{e_F} \in E/\text{gin}_{<\text{rlex}}(J) : \deg \overline{e_F} = j, \min F \geq i+1, \overline{e_F} \neq 0, \overline{e_i e_F} = 0\}|.$$

Here $\overline{e_F}$ denotes the projection of $e_F \in E$ on $E/\text{gin}_{<\text{rlex}}(J)$.

Proof. Since $\alpha_{i,j}(E/J) = \alpha_{i,j}(E/\text{gin}_{<\text{rlex}}(J))$ we may assume that $J = \text{gin}_{<\text{rlex}}(J)$. Then $\alpha_{i,j}(E/J)$ can be computed using the sequence e_1, \dots, e_n , i.e.,

$$\alpha_{i,j}(E/J) = H^j(E/J + (e_1, \dots, e_{i-1}), e_i) = \left(\frac{(J + (e_1, \dots, e_{i-1})) : e_i}{J + (e_1, \dots, e_i)} \right)_j.$$

Thus

$$\alpha_{i,j}(E/J) = |\{e_F : \deg e_F = j, e_F \in (J + (e_1, \dots, e_{i-1})) : e_i, e_F \notin J + (e_1, \dots, e_i)\}|.$$

Let $e_F \in (J + (e_1, \dots, e_{i-1})) : e_i$ of degree j . Then $e_i e_F \in J + (e_1, \dots, e_{i-1})$. Since J is a monomial ideal, either $e_i e_F \in (e_1, \dots, e_{i-1})$ or $e_i e_F \in J$. In the first case it follows that $e_F \in (e_1, \dots, e_{i-1})$ so that we do not need to count it. In the second case, $e_F \notin J + (e_1, \dots, e_i)$ is equivalent to $e_F \notin J$ and $e_F \notin (e_1, \dots, e_i)$ or equivalently $\overline{e_F} \neq 0$ and $\min F \geq i+1$. \square

In the special case of the exterior Stanley-Reisner ring of a simplicial complex Δ , Theorem 3.9 yields the following combinatorial description of the exterior generic annihilator numbers.

Corollary 3.10. *Let Δ be a simplicial complex and let Δ^e be its exterior shifting. Then*

$$\alpha_{i,j}(K\{\Delta\}) = |\{F \in \Delta^e : |F| = j, [i] \cap F = \emptyset, F \cup \{i\} \notin \Delta^e\}|.$$

Using this description we are able to express the symmetric Betti numbers of the Stanley-Reisner ring of the exterior shifting of a simplicial complex as a linear combination of certain generic annihilator numbers. One of the key ingredients for the proof of the formula aforementioned is the well-known Eliahou-Kervaire formula for the symmetric Betti numbers, for squarefree stable ideals. We recall this formula together with some definitions before giving our result. For a monomial ideal I in the polynomial ring or in the exterior algebra we denote by $G(I)$ the unique minimal monomial generating set of I . Let further denote $G(I)_j$ the set of those monomials in $G(I)$ which are of degree j for $j \in \mathbb{Z}$. The Eliahou-Kervaire formula gives an explicit way of how one can compute the symmetric Betti numbers of a squarefree stable ideal.

Proposition 3.11. [AHH98, Corollary 2.3] *Let $I \subseteq S$ be a squarefree stable ideal. Then*

$$\beta_{i,i+j}(S/I) = \sum_{u \in G(I)_{j+1}} \binom{n - \min(u) - j}{i-1}.$$

For a simplicial complex Δ , as Δ^e is shifted, the Stanley-Reisner ideal I_{Δ^e} of the exterior shifting is a squarefree stable ideal. We need this property in the following.

Our result expresses the annihilator numbers in terms of the minimal generators of I_{Δ^e} in the polynomial ring or of J_{Δ^e} in the exterior algebra.

Proposition 3.12. *Let Δ be a simplicial complex and Δ^e be its exterior shifting. Then*

$$\alpha_{l,j}(E/J_{\Delta}) = |\{u \in G(I_{\Delta^e})_{j+1} : \min(u) = l\}| = |\{u \in G(J_{\Delta^e})_{j+1} : \min(u) = l\}|.$$

In particular,

$$\beta_{i,i+j}^S(K[\Delta^e]) = \sum_{l=1}^n \binom{n-l-j}{i-1} \alpha_{l,j}(K\{\Delta\}).$$

Proof. As shown in Corollary 3.10 the number $\alpha_{l,j}(E/J_{\Delta})$ counts the cardinality of the set

$$\mathcal{A} = \{A \in \Delta^e : |A| = j, [l] \cap A = \emptyset, A \cup \{l\} \notin \Delta^e\}.$$

On the other hand the minimal generators of I_{Δ^e} or J_{Δ^e} are the monomials corresponding to minimal non-faces of Δ^e , i.e. the elements of $\{u \in G(I_{\Delta^e})_{j+1} : \min(u) = l\}$ are the monomials x_B such that B lies in the set

$$\mathcal{B} = \{B \notin \Delta^e : |B| = j+1, \min(B) = l, \partial(B) \subseteq \Delta^e\},$$

where $\partial(B) = \{F \subset B : F \neq B\}$ denotes the boundary of B .

We show that there is a one-to-one correspondence between \mathcal{A} and \mathcal{B} . Let $B \in \mathcal{B}$. Then $l \in B$ and $A = B \setminus \{l\}$ is an element in \mathcal{A} . Conversely if $A \in \mathcal{A}$ then $B = A \cup \{l\} \in \mathcal{B}$. The only non-trivial point here is to see that the boundary of B is contained in Δ^e . This holds since Δ^e is shifted and $\min(B) = l$.

The statement about the Betti numbers then follows from the Eliahou-Kervaire formula for squarefree stable ideals (Proposition 3.11). \square

The exterior generic annihilator numbers can also be used to compute the Betti numbers over the exterior algebra. This is analogous to a result over the polynomial ring, see [CHH04, Corollary 1.2]. To this end we use the *Cartan-Betti numbers* introduced by Nagel, Römer and Vinai in [NRV08].

Definition 3.13. Let $J \subseteq E$ be a graded ideal and let v_1, \dots, v_n be a basis of E_1 . We set

$$h_{i,j}(r)(v_1, \dots, v_n; E/J) = \dim_K H_i(v_1, \dots, v_r; E/J)_j,$$

where $H_i(v_1, \dots, v_n; E/J)$ denotes the i -th Cartan homology.

Nagel, Römer and Vinai remarked that there exists a non-empty Zariski-open set W such that the $h_{i,j}$ are constant on it. Therefore they define:

Definition 3.14. Let $J \subseteq E$ be a graded ideal and let v_1, \dots, v_n be a basis of E_1 . We set

$$h_{i,j}(r)(E/J) = h_{i,j}(r)(v_1, \dots, v_n; E/J)$$

for $(v_1, \dots, v_n) \in W$ as above and call these numbers the *Cartan-Betti numbers* of E/J .

For $r = n$, we obtain from Proposition 3.3 that the Cartan-Betti numbers of E/J are the usual exterior graded Betti numbers of E/J , i.e., $h_{i,j}(n)(E/J) = \beta_{i,j}^E(E/J)$.

We formulate and prove the following result using the generic annihilator numbers. Plugging in the description of the generic annihilator numbers in terms of the minimal generators of J_{Δ^e} and taking into account that we use the reversed order on $[n]$, our result is the same as [NRV08, Theorem 2.4(i)] which is a direct consequence of the construction of the Cartan homology for stable ideals in [AHH97, Proposition 3.1].

Theorem 3.15. *Let $J \subseteq E$ be a graded ideal. Then*

$$h_{i,i+j}(r) \leq \sum_{k=1}^r \binom{r+i-k-1}{i-1} \alpha_{k,j}(E/J) \quad i \geq 1, j \geq 0$$

and equality holds for all $i \geq 1$ and $1 \leq r \leq n$ if and only if J is componentwise linear.

Proof. Let v_1, \dots, v_n be a sequence of linear forms that can be used to compute the exterior generic annihilator numbers and the Cartan-Betti numbers of E/J as well. (Such a sequence exists as both conditions are Zariski-open and the intersection of two non-empty Zariski-open set remains non-empty.) Set $\alpha_{i,j} = \alpha_{i,j}(E/J)$ and

$$A_i = \text{Ker} \left(E / (J + (v_1, \dots, v_i)) \xrightarrow{\cdot v_i} E / (J + (v_1, \dots, v_{i-1})) \right)$$

for $1 \leq i \leq n$. Then A_i is a graded E -module and the K -vector space dimension of the j -th graded piece equals $\alpha_{i,j}(E/J)$. The above map occurs in the long exact sequence of Cartan homologies (see Proposition 3.2) since the 0-th Cartan homology is

$$H_0(v_1, \dots, v_r; E/J) = E / (J + (v_1, \dots, v_r)).$$

Thus for $i = 1$ and $r = 1$ we obtain from the long exact Cartan homology sequence the following exact sequence

$$H_1(1)(-1)_{j+1} \rightarrow H_1(0)_{j+1} \rightarrow H_1(1)_{j+1} \rightarrow A_1(-1)_{j+1} \rightarrow 0.$$

Since $H_i(0) = 0$ for $i \geq 1$ this yields

$$h_{1,j+1}(1) = \alpha_{1,j}.$$

For $r \geq 1$ we have the exact sequence

$$H_1(r+1)(-1)_{j+1} \rightarrow H_1(r)_{j+1} \rightarrow H_1(r+1)_{j+1} \rightarrow A_{r+1}(-1)_{j+1} \rightarrow 0.$$

From this sequence we conclude by induction hypothesis on r

$$\begin{aligned} h_{1,j+1}(r+1) &\leq \alpha_{r+1,j} + h_{1,j+1}(r) \\ &\leq \alpha_{r+1,j} + \sum_{k=1}^r \binom{r-k}{0} \alpha_{k,j} \\ &= \sum_{k=1}^{r+1} \alpha_{k,j}. \end{aligned}$$

Now let $i > 1$. For $r = 1$ there is the exact sequence

$$H_i(0)_{i+j} \rightarrow H_i(1)_{i+j} \rightarrow H_{i-1}(1)(-1)_{i+j} \rightarrow H_{i-1}(0)_{i+j}.$$

The outer spaces in the sequence are zero, hence

$$h_{i,i+j}(1) = h_{i-1,i+j-1}(1) \leq \alpha_{1,j}$$

by induction hypothesis on i .

Let now $r \geq 1$. There is the exact sequence

$$H_i(r)_{i+j} \rightarrow H_i(r+1)_{i+j} \rightarrow H_{i-1}(r+1)(-1)_{i+j} \rightarrow H_{i-1}(r)_{i+j}.$$

We conclude by induction hypothesis on r and i

$$\begin{aligned} h_{i,i+j}(r+1) &\leq h_{i,i+j}(r) + h_{i-1,i-1+j}(r+1) \\ &\leq \sum_{k=1}^r \binom{r+i-k-1}{i-1} \alpha_{k,j} + \sum_{k=1}^{r+1} \binom{r+1+i-1-k-1}{i-2} \alpha_{k,j} \\ &= \sum_{k=1}^r \left(\binom{r+i-k-1}{i-1} + \binom{r+i-k-1}{i-2} \right) \alpha_{k,j} + \binom{i-2}{i-2} \alpha_{r+1,j} \\ &= \sum_{k=1}^{r+1} \binom{r+i-k}{i-1} \alpha_{k,j}. \end{aligned}$$

The inequalities in the proof are all equalities if and only if the long exact sequence is split exact. In this case the sequence v_1, \dots, v_n is called a *proper sequence* for E/J . In [NRV08, Theorem 2.10] it is shown that this is the case if and only if J is a componentwise linear ideal. \square

4. AN UNEXPECTED BEHAVIOR OF THE GENERIC ANNIHILATOR NUMBERS

A natural question to ask is whether the exterior generic annihilator numbers play a special role among the exterior annihilator numbers of E/J with respect to a certain sequence. Herzog posed the question if they are the minimal ones among all the annihilator numbers. In the attempt of proving this conjecture it turned out to be wrong. In order to clarify this unexpected result we do not only give a counterexample of the conjecture but we also give a sketch of the original idea of the proof and explain how we came up with the example. This also gives a hint at how to construct further counterexamples. After some slight changes our example also serves as a counterexample of the corresponding conjecture for the symmetric generic annihilator numbers. For the sake of completeness we first state the conjecture.

Conjecture 4.1. Let $J \subseteq E$ be a graded ideal. For any basis v_1, \dots, v_n of E_1 it holds that

$$\alpha_{i,j}(E/J) \leq \alpha_{i,j}(v_1, \dots, v_n; E/J)$$

for $1 \leq i \leq n$ and $j \geq 0$.

Thus, our aim was to prove that the annihilator numbers are minimal on a non-empty Zariski-open set. For $i = 1$ this is known to be true. Just take the non-empty Zariski-open set such that the ranks of the matrices of the maps of the complex

$$(M, v) : \quad \dots \longrightarrow M_{j-1} \xrightarrow{\cdot v} M_j \xrightarrow{\cdot v} M_{j+1} \longrightarrow \dots$$

are maximal (note that $M_j = 0$ for almost all j). To prove this for longer sequences we tried to show that the sets

$$U_{i,j} = \left\{ (v_1, \dots, v_n) \subseteq E_1 \text{ basis} : \begin{array}{l} \alpha_{i,j}(v_1, \dots, v_n; E/J) \leq \alpha_{i,j}(w_1, \dots, w_n; E/J) \\ \text{for any basis } (w_1, \dots, w_n) \subseteq E_1 \end{array} \right\}$$

were non-empty Zariski-open sets for $1 \leq i \leq n$, $0 \leq j \leq n$. The intersection of those sets would have been a non-empty Zariski-open set having the required property (Note that only finitely many sets are intersected.). In order to compute a certain annihilator number $\alpha_{i,j}(v_1, \dots, v_n; E/J)$ we used the exact sequence

$$\begin{aligned} 0 \longrightarrow H^j(E/(J + (v_1, \dots, v_{i-1})), v_i) &\longrightarrow (E/(J + (v_1, \dots, v_i)))_j \\ &\xrightarrow{\cdot v_i} (E/(J + (v_1, \dots, v_{i-1})))_{j+1} \longrightarrow (E/(J + (v_1, \dots, v_i)))_{j+1} \longrightarrow 0, \end{aligned}$$

which yields

$$\begin{aligned} \alpha_{i,j}(v_1, \dots, v_n; E/J) &= \dim_K(E/(J + (v_1, \dots, v_i)))_j \\ &\quad - (\dim_K(E/(J + (v_1, \dots, v_{i-1})))_{j+1} - \dim_K(E/(J + (v_1, \dots, v_i)))_{j+1}). \end{aligned}$$

One can show that for a generic basis $\{v_1, \dots, v_n\} \subseteq E_1$ each of the vector space dimensions on the right-hand side of the above equality is minimized. However, being the left-hand side of the above equality an alternating sum of those three dimensions there is no reason to expect it to be minimized by a generic basis.

Example 4.2. Let $1 \leq i, j \leq n$ and let $J = (e_{l_1} \cdot \dots \cdot e_{l_{j+1}} : i \leq l_1 < l_2 < \dots < l_{j+1} \leq n) \subseteq E$ be a graded ideal. By construction, J is strongly stable and it therefore holds that $\text{gin}_{<\text{rlex}}(J) = J$. Then

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(e_1, \dots, e_n; E/\text{gin}_{<\text{rlex}}(J)) = \alpha_{i,j}(e_1, \dots, e_n; E/J),$$

i.e., we can use the sequence e_1, \dots, e_n to compute the generic annihilator numbers of E/J .

From the exact sequence

$$\begin{aligned} 0 \longrightarrow H^j(E/(J + (e_1, \dots, e_{i-1})), e_i) &\longrightarrow (E/(J + (e_1, \dots, e_i)))_j \\ &\xrightarrow{\cdot e_i} (E/(J + (e_1, \dots, e_{i-1})))_{j+1} \longrightarrow (E/(J + (e_1, \dots, e_i)))_{j+1} \longrightarrow 0 \end{aligned}$$

we deduce

$$\begin{aligned} \alpha_{i,j} &= \dim_K(E/(J + (e_1, \dots, e_i)))_j \\ &\quad - (\dim_K(E/(J + (e_1, \dots, e_{i-1})))_{j+1} - \dim_K(E/(J + (e_1, \dots, e_i)))_{j+1}). \end{aligned}$$

In the following we consider the sequence $\mathbf{e} = e_1, \dots, e_{i-2}, e_i, e_{i-1}, e_{i+1}, \dots, e_n$ and compute the exterior annihilator numbers of E/J with respect to this sequence. As before, we have the exact sequence

$$\begin{aligned} 0 \longrightarrow H^j(E/(J + (e_1, \dots, e_{i-2}, e_i)), e_{i-1}) &\longrightarrow (E/(J + (e_1, \dots, e_i)))_j \\ &\xrightarrow{\cdot e_{i-1}} (E/(J + (e_1, \dots, e_{i-2}, e_i)))_{j+1} \longrightarrow (E/(J + (e_1, \dots, e_i)))_{j+1} \longrightarrow 0 \end{aligned}$$

which leads to

$$\begin{aligned} \alpha_{i,j}(\mathbf{e}; E/J) &= \dim_K(E/(J + (e_1, \dots, e_i)))_j \\ &\quad - (\dim_K(E/(J + (e_1, \dots, e_{i-2}, e_i)))_{j+1} - \dim_K(E/(J + (e_1, \dots, e_i)))_{j+1}). \end{aligned}$$

Our aim is to show that $\alpha_{i,j}(E/J) > \alpha_{i,j}(\mathbf{e}; E/J)$. We therefore need to show that

$$\dim_K(E/(J + (e_1, \dots, e_{i-2}, e_i)))_{j+1} > \dim_K(E/(J + (e_1, \dots, e_{i-2}, e_{i-1})))_{j+1}.$$

Let $m = e_{l_1} \cdots e_{l_{j+1}} \in E_{j+1}$ with $l_1 < \dots < l_{j+1}$. If $l_1 \leq i-1$, it already holds that $m \in (e_1, \dots, e_{i-1})_{j+1} \subseteq (J + (e_1, \dots, e_{i-1}))_{j+1}$. If $l_1 \geq i$, we have $i \leq l_1 < \dots < l_{j+1}$ and therefore $m \in J_{j+1} \subseteq (J + (e_1, \dots, e_{i-1}))_{j+1}$. Thus, $m \in (J + (e_1, \dots, e_{i-1}))_{j+1}$ in either case and therefore $(J + (e_1, \dots, e_{i-1}))_{j+1} = E_{j+1}$ and so $\dim_K(E/(J + (e_1, \dots, e_{i-1})))_{j+1} = 0$.

Consider now $\tilde{m} = e_{i-1}e_{i+1} \cdots e_{i+j} \in E_{j+1}$. By definition, it holds that $\tilde{m} \notin J$ and $\tilde{m} \notin (e_1, \dots, e_{i-2}, e_i)$. Since J is a monomial ideal this implies $\tilde{m} \notin (J + (e_1, \dots, e_{i-2}, e_i))_{j+1}$. We therefore get $\dim_K(E/(J + (e_1, \dots, e_{i-2}, e_i)))_{j+1} > 0$. This finally shows

$$\alpha_{i,j}(E/J) > \alpha_{i,j}(\mathbf{e}; E/J).$$

We also compute the annihilator numbers one step before w.r.t. this two sequences, i.e., $\alpha_{i-1,j}(E/J)$ and $\alpha_{i-1,j}(\mathbf{e}; E/J)$ to show that those numbers are related to each other the other way round, i.e., we have

$$(4) \quad \alpha_{i-1,j}(E/J) < \alpha_{i-1,j}(\mathbf{e}; E/J).$$

This suggests that, to have a chance to become smaller than the generic numbers, the annihilator numbers with respect to \mathbf{e} first have to become “worse”, i.e., greater. Similar to the i -th annihilator numbers of E/J we can compute the $(i-1)$ -st annihilator numbers using the exact sequence. We therefore get

$$\begin{aligned} \alpha_{i-1,j}(E/J) &= \dim_K(E/(J + (e_1, \dots, e_{i-1})))_j \\ &\quad - (\dim_K(E/(J + (e_1, \dots, e_{i-2})))_{j+1} - \dim_K(E/(J + (e_1, \dots, e_{i-1})))_{j+1}) \end{aligned}$$

for the $(i-1)$ -st generic annihilator of E/J in degree j . In the same way, we obtain

$$\begin{aligned} \alpha_{i-1,j}(\mathbf{e}; E/J) &= \dim_K(E/(J + (e_1, \dots, e_{i-2}, e_i)))_j \\ &\quad - (\dim_K(E/(J + (e_1, \dots, e_{i-2})))_{j+1} - \dim_K(E/(J + (e_1, \dots, e_{i-2}, e_i)))_{j+1}) \end{aligned}$$

for the $(i-1)$ -st exterior annihilator number of E/J in degree j with respect to the sequence \mathbf{e} . Since J is generated by monomials of degree strictly larger than j it holds that

$$\dim_K(E/(J + (e_1, \dots, e_{i-1})))_j = \dim_K(E/(e_1, \dots, e_{i-1}))_j$$

and

$$\dim_K(E/(J + (e_1, \dots, e_{i-2}, e_i)))_j = \dim_K(E/(e_1, \dots, e_{i-2}, e_i))_j.$$

Since

$$\dim_K(E/(J + (e_1, \dots, e_{i-1})))_j = \dim_K(E/(J + (e_1, \dots, e_{i-2}, e_i)))_j,$$

in order to show (4) we only need to prove that

$$\dim_K(E/(J + (e_1, \dots, e_{i-1})))_{j+1} < \dim_K(E/(J + (e_1, \dots, e_{i-2}, e_i)))_{j+1}.$$

This follows from

$$(5) \quad (J + (e_1, \dots, e_{i-1}))_{j+1} \supsetneq (J + (e_1, \dots, e_{i-2}, e_i))_{j+1}.$$

To show (5) let $m = e_{l_1} \cdots e_{l_{j+1}} \in (J + (e_1, \dots, e_{i-2}, e_i))_{j+1}$ with $l_1 < \dots < l_{j+1}$. If $l_1 \geq i$ it follows that $m \in J_{j+1} \subseteq (J + (e_1, \dots, e_{i-1}))_{j+1}$. If $l_1 \leq i-1$ it already holds

that $e_{l_1} \in (e_1, \dots, e_{i-1})$ and thus $m \in (J + (e_1, \dots, e_{i-1}))_{j+1}$. Since $e_{i-1}e_{i+1} \cdots e_{i+j} \in (J + (e_1, \dots, e_{i-1}))_{j+1}$ but $e_{i-1}e_{i+1} \cdots e_{i+j} \notin (J + (e_1, \dots, e_{i-2}, e_i))_{j+1}$ we obtain (5).

After slight modifications to the above example (a kind of “de-polarization”) we get a counterexample of the conjecture that the symmetric generic annihilator numbers are the minimal ones among all the annihilator numbers with respect to a sequence. Before treating this very example in more detail we recall the precise definition of the symmetric annihilator numbers. Let v_1, \dots, v_n be a K -basis of S_1 and let M be a finitely generated graded S -module. We set $A_i(v_1, \dots, v_n; M) = 0 :_{M/(v_1, \dots, v_{i-1})M} v_i$ and call the numbers

$$\alpha_{i,j}(v_1, \dots, v_n; M) = \begin{cases} \dim_K A_i(v_1, \dots, v_n; M)_j, & \text{if } 1 \leq i \leq n \\ \beta_{0,j}(M), & \text{if } i = n+1 \end{cases}$$

the *symmetric annihilator numbers* of M with respect to the sequence v_1, \dots, v_n . Let now $v \in S_1$ be a linear form. If the multiplication map $M_{i-1} \rightarrow M_i$ is injective for all $i \gg 0$, i.e., if only finitely many graded components of $0 :_M v$ are non-zero, v is called an *almost regular element* on M . A sequence v_1, \dots, v_r is an *almost regular sequence* on M if v_i is almost regular on $M/(v_1, \dots, v_{i-1})M$ for $1 \leq i \leq r$. Note that for an almost regular sequence only finitely many annihilator numbers are non-zero.

Herzog and Hibi proved in [HH08] that the set of almost regular sequences on M is a non-empty Zariski-open set. They further show that v_1, \dots, v_t is an M -regular sequence if v_1, \dots, v_n is an almost regular sequence on M and $\text{depth}_S(M) = t$. Hence this gives a proof of the following well-known fact.

Proposition 4.3. *Let M be a finitely generated graded S -module and let $\text{depth}_S(M) = t$. There exists a Zariski-open set $U \subseteq GL_n(K)$ such that $\gamma_{1,1}x_1 + \dots + \gamma_{n,1}x_n, \dots, \gamma_{1,t}x_1 + \dots + \gamma_{n,t}x_n$ is an M -regular sequence for all $\gamma = (\gamma_{i,j})_{1 \leq i,j \leq n} \in U$.*

As for the exterior annihilator numbers Herzog and Hibi show that the symmetric annihilator numbers with respect to a sequence coincide when choosing the sequence from a certain non-empty Zariski-open set. The proof is very similar to the proof of Theorem 3.5, the corresponding statement over the exterior algebra.

Theorem 4.4. [HH08] *Let $I \subseteq S$ be a graded ideal. Then there exists a non-empty Zariski-open set $U \subseteq GL_n(K)$ such that $\gamma(x) = (\sum_{i=1}^n \gamma_{i,1}x_i, \dots, \sum_{i=1}^n \gamma_{i,n}x_i)$ is almost regular for all $\gamma = (\gamma_{i,j})_{1 \leq i,j \leq n} \in U$. Moreover,*

$$\alpha_{i,j}(\gamma(x); S/I) = \alpha_{i,j}(x_1, \dots, x_n; S/\text{gin}_{<\text{rlex}}(I))$$

for all $\gamma \in U$.

This gives rise to the definition of the symmetric generic annihilator numbers. The numbers

$$\alpha_{i,j}(S/I) = \alpha_{i,j}(x_1, \dots, x_n; S/\text{gin}_{<\text{rlex}}(I))$$

are called the *symmetric generic annihilator numbers* of S/I . As in the case of the exterior generic annihilator numbers one can wonder if the symmetric generic annihilator numbers are the minimal ones among the symmetric annihilator numbers with respect to a sequence. As already mentioned, this is not the case.

Example 4.5. Let $1 \leq i \leq j \leq n$ and let $I = (x_{l_1} \cdots x_{l_{j+1}} : i \leq l_1 \leq \dots \leq l_{j+1}) \subseteq S$ be a graded ideal. By construction I is strongly stable and it therefore holds that $\text{gin}_{<\text{rlex}}(I) = I$. Analogously to Example 4.2 we can use the sequence x_1, \dots, x_n to compute the symmetric generic annihilator numbers of S/I . From the exact sequence

$$0 \longrightarrow A_i(x_1, \dots, x_n; S/I)_j \longrightarrow (S/(I + (x_1, \dots, x_{i-1})))_j \\ \xrightarrow{\cdot x_i} (S/(I + (x_1, \dots, x_{i-1})))_{j+1} \longrightarrow (S/(I + (x_1, \dots, x_i)))_{j+1} \longrightarrow 0$$

we deduce

$$\alpha_{i,j}(S/I) = \dim_K(S/(I + (x_1, \dots, x_{i-1})))_j \\ - (\dim_K(S/(I + (x_1, \dots, x_{i-1})))_{j+1} - \dim_K(S/(I + (x_1, \dots, x_i)))_{j+1}).$$

With the same exact sequence for the sequence $\mathbf{x} = x_1, \dots, x_{i-2}, x_i, x_{i-1}, x_{i+1}, \dots, x_n$ we get

$$\alpha_{i,j}(\mathbf{x}; S/I) = \dim_K(S/(I + (x_1, \dots, x_{i-2}, x_i)))_j \\ - (\dim_K(S/(I + (x_1, \dots, x_{i-2}, x_i)))_{j+1} - \dim_K(S/(I + (x_1, \dots, x_i)))_{j+1}).$$

Since I is generated in degree $j+1$ it holds that

$$\dim_K(S/(I + (x_1, \dots, x_{i-1})))_j = \dim_K(S/(x_1, \dots, x_{i-1}))_j \\ = \dim_K(S/(x_1, \dots, x_{i-2}, x_i))_j \\ = \dim_K(S/(I + (x_1, \dots, x_{i-2}, x_i)))_j.$$

One easily shows that $(I + (x_1, \dots, x_{i-2}, x_i))_{j+1} \subsetneq (I + (x_1, \dots, x_{i-1}))_{j+1}$. This implies

$$(6) \quad \dim_K(S/(I + (x_1, \dots, x_{i-1})))_{j+1} < \dim_K(S/(I + (x_1, \dots, x_{i-2}, x_i)))_{j+1}.$$

As in the exterior case we thus obtain

$$\alpha_{i,j}(S/I) > \alpha_{i,j}(\mathbf{x}; S/I).$$

We now compute the $(i-1)$ -st annihilator numbers in degree j and see what happens in this case. The same arguments as before show that

$$\alpha_{i-1,j}(S/I) = \dim_K(S/(I + (x_1, \dots, x_{i-2})))_j \\ - (\dim_K(S/(I + (x_1, \dots, x_{i-2})))_{j+1} - \dim_K(S/(I + (x_1, \dots, x_{i-1})))_{j+1})$$

and

$$\alpha_{i-1,j}(\mathbf{x}; S/I) = \dim_K(S/(I + (x_1, \dots, x_{i-2})))_j \\ - (\dim_K(S/(I + (x_1, \dots, x_{i-2})))_{j+1} - \dim_K(S/(I + (x_1, \dots, x_{i-2}, x_i)))_{j+1}).$$

Using Equation (6) we thus obtain

$$\alpha_{i-1,j}(S/I) < \alpha_{i-1,j}(\mathbf{x}; S/I).$$

Example 4.2 and 4.5 in particular show that changing the order of the elements of a sequence might change the annihilator numbers. However, when taking the sequence from a certain non-empty Zariski-open set the order of the elements does not matter.

Theorem 4.6. *Let $J \subseteq E$ be a graded ideal. If K has enough algebraically independent elements over its base field, then there exists a non-empty Zariski-open set $V \subseteq GL_n(K)$ such that*

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(\gamma(e_{\sigma(1)}), \dots, \gamma(e_{\sigma(n)}); E/J)$$

for all $\gamma \in V$ and all $\sigma \in S_n$, where S_n denotes the symmetric group on $[n]$.

Proof. If K has enough algebraically independent elements over its base field, then the subset $V \subseteq GL_n(K)$ consisting of matrices with algebraically independent entries is non-empty and Zariski-open. Furthermore, it is invariant under row permutations, column permutations and also under taking inverses. We show that V is as required.

Consider $\gamma \in V$, $\sigma \in S_n$ and set $\gamma_i = \gamma(e_i)$. The proof of Theorem 3.5 shows that it is enough to prove

$$\dim_K (E / (\text{gin}_{<\text{rlex}}(J) + (e_1, \dots, e_i)))_j = \dim_K (E / (J + (\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(i)})))_j$$

for $i = 1, \dots, n$ and $j \in \mathbb{Z}$.

Since $\text{gin}_{<\text{rlex}}(J) = \text{in}_{<\text{rlex}}(\gamma^{-1}(J))$ it holds that

$$\dim_K (E / (\text{gin}_{<\text{rlex}}(J) + (e_1, \dots, e_i)))_j = \dim_K (E / (J + (\gamma_1, \dots, \gamma_i)))_j.$$

The permutation lemma on page 288 in [BK88] (it is only formulated for monomial ideals, but the proof is valid also for arbitrary graded ideals) implies that

$$\dim_K (E / (J + (\gamma_1, \dots, \gamma_i)))_j = \dim_K (E / (J + (\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(i)})))_j.$$

Both equations together conclude the proof. \square

It would be nice to know if one can drop the assumption on K in the above Theorem 4.6.

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REFERENCES

- [AAH00] A. Aramova, L. L. Avramov, and J. Herzog, *Resolutions of monomial ideals and cohomology over exterior algebras*, Trans. Am. Math. Soc. **352** (2000), no. 2, 579–594.
- [AH00] A. Aramova and J. Herzog, *Almost regular sequences and Betti numbers*, Am. J. Math. **122** (2000), no. 4, 689–719.
- [AHH97] A. Aramova, J. Herzog, and T. Hibi, *Gotzmann Theorems for Exterior Algebras and Combinatorics*, J. Algebra **191** (1997), 174–211.
- [AHH98] ———, *Squarefree lexsegment ideals*, Math. Z. **228** (1998), 353–378.
- [AHH00] ———, *Shifting Operations and Graded Betti Numbers*, J. Alg. Comb. **12** (2000), 207–222.
- [BH98] W. Bruns and J. Herzog, *Cohen-Macaulay rings. Rev. ed.*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, 1998.

- [BK88] A. Björner and G. Kalai, *An extended Euler-Poincaré theorem*, Acta Math. **161** (1988), no. 3-4.
- [BS87] D. Bayer and M. Stillman, *A criterion for detecting m -regularity*, Invent. Math. **87** (1987), 1–11.
- [CHH04] A. Conca, J. Herzog, and T. Hibi, *Rigid resolutions and big Betti numbers*, Comment. Math. Helv. **79** (2004), 826–839.
- [Eis04] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer, 2004.
- [HH08] J. Herzog and T. Hibi, *Monomials*, Book manuscript, 2008.
- [HT99] J. Herzog and N. Terai, *Stable properties of algebraic shifting*, Result in Math. **35** (1999), 260–265.
- [Kal01] G. Kalai, *Algebraic shifting*, Computational Commutative Algebra and Combinatorics. Adv. Stud. Pure Math. **33** (2001), 121–163.
- [KR09] G. Kämpf and T. Römer, *Homological properties of Orlik-Solomon algebras*, Manuscr. Math. **129** (2009), 181–210.
- [KW08] M. Kubitzke and V. Welker, *The Multiplicity Conjecture for Barycentric Subdivisions*, Commun. Algebra **36** (2008), no. 11.
- [NRV08] U. Nagel, T. Römer, and N. P. Vinai, *Algebraic shifting and exterior and symmetric algebra methods*, Commun. Algebra **36** (2008), no. 1, 208–231.
- [Röm01] T. Römer, *Generalized Alexander Duality and Applications*, Osaka J. Math. **38** (2001), 469–485.
- [Swa06] E. Swartz, *g -elements, finite buildings and higher Cohen-Macaulay connectivity*, J. Comb. Theory, Ser. A **113** (2006), no. 7, 1305–1320.
- [Tru87] N. V. Trung, *Reduction exponent and degree bound for the defining equations of graded rings*, Proc. Amer. Math. Soc. **101** (1987), 229–236.
- [Yan00] K. Yanagawa, *Alexander Duality for Stanley-Reisner Rings and Squarefree \mathbb{N}^n -Graded Modules*, J. Algebra **225** (2000), 630–645.

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